

# ROTA-BAXTER COALGEBRAS

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**ABSTRACT.** We introduce the notion of Rota-Baxter coalgebra which can be viewed as the dual notion of Rota-Baxter algebra. We provide various concrete examples and establish some basic properties of this new object.

## 1. INTRODUCTION

A Rota-Baxter algebra of weight  $\lambda$  is a pair  $(R, P)$  consisting of an associative algebra  $R$  and an endomorphism  $P$  of  $R$  such that

$$P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy), \quad \forall x, y \in R.$$

They are first initiated by the work of G. Baxter on probability theory ([4]) and formulated formally by Rota ([22]) in 1960s. Later, even though some famous mathematicians, such as Cartier, studied these new subjects, but they did not draw people's attention extensively. Things are changed in 2000 after the works [14] and [16]. From then on, these algebras become popular. Nowadays, except for their own interests (see e.g., [5]), they have many applications in other areas of mathematics and mathematical physics, such as combinatorics ([9]), Loday type algebras ([8], [10]), pre-Lie and pre-Poisson algebras ([20], [2], [1]), multiple zeta values ([11], [17]), and Connes-Kreimer renormalization theory in quantum field theory ([12], [13]). For a detailed description of the theory of Rota-Baxter algebra, we refer the reader to the clearly written book [15].

Stimulating by the theory of coalgebras, it seems reasonable to consider the dual theory of Rota-Baxter algebras. The significance of studying dual theories is that they do not only bring us to new phenomena but also help us to understand the original subjects much better. The purpose of this paper is intended to investigate the dual theory of Rota-Baxter algebras. We first define the notion of Rota-Baxter coalgebras. It can be viewed as the dual version of Rota-Baxter algebras. Since abundant examples show the vitality of a new subject, we provide various examples of Rota-Baxter coalgebras, including constructions by group-like elements and by smash coproduct. It is a direct and easy verification that the dual coalgebra of a Rota-Baxter algebra together with the linear dual of the Rota-Baxter operator is a Rota-Baxter coalgebra. But the converse is not true in general. One of the problem is that the linear dual of an algebra is not a coalgebra unless it is finite dimensional. In order to overcome this difficulty, we restrict our attention to the finite dual of an algebra. To make the converse statement hold, we also have to impose some relevant conditions on the original algebra structure that relate to the so-called double product. We study the relation between Rota-Baxter coalgebras with counit

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and those without counit in the idempotent case as well. On the other hand, we want to establish the properties of Rota-Baxter coalgebras having dual versions in Rota-Baxter algebras. For example, we construct the double coproduct which is dual to the double product, and we provide necessary and sufficient conditions for a Rota-Baxter coalgebra being idempotent.

This paper is organized as follows. In Section 2, we introduce the notion of Rota-Baxter coalgebra and provide several concrete examples. In Section 3, we establish various properties of Rota-Baxter coalgebras.

## 2. ROTA-BAXTER COALGEBRAS

Throughout this paper, we fix a ground field  $\mathbb{K}$  of characteristic 0 and assume that all vector spaces, algebras, coalgebras and tensor products are defined over  $\mathbb{K}$ .

We adopt Sweedler's notation for coalgebras and comodules. Let  $(C, \Delta, \varepsilon)$  be a coalgebra. For any  $c \in C$ , we denote

$$\Delta(c) = \sum c_{(1)} \otimes c_{(2)}.$$

Let  $(M, \delta)$  be a left  $C$ -comodule. For any  $m \in M$ , we denote

$$\delta(m) = \sum m_{(-1)} \otimes m_{(0)}.$$

**Definition 2.1.** A *Rota-Baxter coalgebra* of weight  $\lambda$  is a coalgebra  $(C, \Delta, \varepsilon)$  equipped with an operator  $P$  such that

$$(1) \quad (P \otimes P)\Delta = (\text{id} \otimes P)\Delta P + (P \otimes \text{id})\Delta P + \lambda \Delta P.$$

In Sweedler's notation, this means

$$\begin{aligned} & \sum P(c_{(1)}) \otimes P(c_{(2)}) \\ &= \sum P(c)_{(1)} \otimes P(P(c)_{(2)}) + \sum P(P(c)_{(1)}) \otimes P(c)_{(2)} + \lambda \sum P(c)_{(1)} \otimes P(c)_{(2)}. \end{aligned}$$

We always use the quadruple  $(C, \Delta, \varepsilon, P)$  to denote a Rota-Baxter coalgebra. If  $(C, \Delta)$  is a noncounitary coalgebra equipped with an endomorphism  $P$  which satisfies equation (1) above, we call the triple  $(C, \Delta, P)$  a *noncounitary Rota-Baxter coalgebra*.

A Rota-Baxter coalgebra is called *idempotent* if the Rota-Baxter operator  $P$  satisfies  $P^2 = P$ .

*Remark 2.2.* (1) If  $(C, \Delta, P)$  is a (noncounitary) Rota-Baxter coalgebra of weight  $\lambda \neq 0$ , then it is a direct verification that  $(C, \Delta, \mu\lambda^{-1}P)$  is a (noncounitary) Rota-Baxter coalgebra of weight  $\mu$ .

(2) If  $(C, \Delta, \varepsilon, P)$  is a Rota-Baxter coalgebra, then its dual  $(C^*, \Delta^*, \varepsilon^*, P^*)$  is a unitary Rota-Baxter algebra. Similarly, the dual of a noncounitary Rota-Baxter coalgebra is a nonunitary Rota-Baxter algebra.

**Example 2.3.** Let  $(C, \Delta, \varepsilon)$  be a coalgebra. Assume that  $P : C \rightarrow C$  is a linear map such that  $\Delta P = (P \otimes P)\Delta$  and  $P^2 = P$ . Then  $(C, \Delta, \varepsilon, P)$  is an idempotent Rota-Baxter coalgebra of weight  $-1$ . Indeed, for any  $c \in C$ , we have

$$\begin{aligned} & (P \otimes \text{id})\Delta P(c) + (\text{id} \otimes P)\Delta P(c) - \Delta P(c) \\ &= \sum P^2(c_{(1)}) \otimes P(c_{(2)}) + \sum P(c_{(1)}) \otimes P^2(c_{(2)}) - \sum P(c_{(1)}) \otimes P(c_{(2)}) \\ &= \sum P(c_{(1)}) \otimes P(c_{(2)}) + \sum P(c_{(1)}) \otimes P(c_{(2)}) - \sum P(c_{(1)}) \otimes P(c_{(2)}) \end{aligned}$$

$$\begin{aligned}
 &= \sum P(c_{(1)}) \otimes P(c_{(2)}) \\
 &= (P \otimes P)\Delta(c).
 \end{aligned}$$

Given a group-like element  $g \in C$ , we define  $P_g : C \rightarrow C$  by  $P_g(c) = \varepsilon(c)g$  for any  $c \in C$ . Since  $\Delta(g) = g \otimes g$  and  $\varepsilon(g) = 1$ , we have

$$\begin{aligned}
 \Delta P_g(c) &= \Delta(\varepsilon(c)g) \\
 &= \varepsilon(c)g \otimes g \\
 &= \varepsilon\left(\sum \varepsilon(c_{(1)})c_{(2)}\right)g \otimes g \\
 &= \sum \varepsilon(c_{(1)})g \otimes \varepsilon(c_{(2)})g \\
 &= \sum P_g(c_{(1)}) \otimes P_g(c_{(2)}) \\
 &= (P_g \otimes P_g)\Delta(c),
 \end{aligned}$$

and

$$\begin{aligned}
 P_g^2(c) &= P_g(\varepsilon(c)g) \\
 &= \varepsilon(c)\varepsilon(g)g \\
 &= \varepsilon(c)g \\
 &= P_g(c).
 \end{aligned}$$

Hence  $(C, \Delta, \varepsilon, P_g)$  is an idempotent Rota-Baxter coalgebra of weight  $-1$ .

**Example 2.4.** Let  $(H, \Delta, \varepsilon, S)$  be a Hopf algebra. Define the left  $H$ -comodule structure  $\delta$  on  $H$  by  $\delta(h) = \sum h_{(1)}S(h_{(3)}) \otimes h_{(2)}$ . Then  $H$  is a comodule-coalgebra. So we have the smash coalgebra structure on  $H \otimes H$  given by

$$\Delta_s(a \otimes b) = \sum a_{(1)} \otimes a_{(2)}S(a_{(4)})b_{(1)} \otimes a_{(3)} \otimes b_{(2)},$$

and

$$\varepsilon_s(a \otimes b) = \varepsilon(a)\varepsilon(b).$$

If we define  $P_1(a \otimes b) = a \otimes \varepsilon(b)1_H$ , then  $(H \otimes H, \Delta_s, \varepsilon_s, P_1)$  is an idempotent Rota-Baxter coalgebra. Indeed, we have

$$\begin{aligned}
 &(\text{id} \otimes P_1)\Delta_s P_1(a \otimes b) + (P_1 \otimes \text{id})\Delta_s P_1(a \otimes b) - \Delta_s P_1(a \otimes b) \\
 &= (\text{id} \otimes P_1)\Delta_s(a \otimes \varepsilon(b)1_H) + (P_1 \otimes \text{id})\Delta_s(a \otimes \varepsilon(b)1_H) - \Delta_s(a \otimes \varepsilon(b)1_H) \\
 &= \varepsilon(b) \sum a_{(1)} \otimes a_{(2)}S(a_{(4)}) \otimes P_1(a_{(3)} \otimes 1_H) \\
 &\quad + \varepsilon(b) \sum P_1(a_{(1)} \otimes a_{(2)}S(a_{(4)})) \otimes a_{(3)} \otimes 1_H \\
 &\quad - \varepsilon(b) \sum a_{(1)} \otimes a_{(2)}S(a_{(4)}) \otimes a_{(3)} \otimes 1_H \\
 &= \varepsilon(b) \sum a_{(1)} \otimes a_{(2)}S(a_{(4)}) \otimes a_{(3)} \otimes 1_H \\
 &\quad + \varepsilon(b) \sum a_{(1)} \otimes \varepsilon(a_{(2)})\varepsilon(a_{(4)})1_H \otimes a_{(3)} \otimes 1_H \\
 &\quad - \varepsilon(b) \sum a_{(1)} \otimes a_{(2)}S(a_{(4)}) \otimes a_{(3)} \otimes 1_H
 \end{aligned}$$

$$\begin{aligned}
&= \varepsilon(b) \sum a_{(1)} \otimes \varepsilon(a_{(2)}) \varepsilon(a_{(4)}) 1_H \otimes a_{(3)} \otimes 1_H \\
&= \sum a_{(1)} \otimes \varepsilon(a_{(2)}) \varepsilon(a_{(4)}) \varepsilon(b_{(1)}) 1_H \otimes a_{(3)} \otimes \varepsilon(b_{(2)}) 1_H \\
&= \sum P_1(a_{(1)} \otimes a_{(2)}) S(a_{(4)}) b_{(1)} \otimes P_1(a_{(3)} \otimes b_{(2)}) \\
&= (P_1 \otimes P_1) \Delta_s(a \otimes b).
\end{aligned}$$

Obviously,  $P_1^2 = P_1$ .

On the other hand, if we define

$$P_2(a \otimes b) = \sum S(a_{(2)}) S(a_{(4)}) b_{(2)} a_{(3)} \otimes S(a_{(1)}) S(a_{(5)}) b_{(1)} b_{(3)},$$

then  $(H \otimes H, \Delta_s, \varepsilon_s, P_2)$  is an idempotent Rota-Baxter coalgebra of weight  $-1$  as well.

The space  $H \otimes H$  is a left  $H$ -module via the diagonal action, i.e.,  $h.(a \otimes b) = \sum h_{(1)}a \otimes h_{(2)}b$ , for any  $a, b, h \in H$ . Then the action of  $P_2$  can be rewritten as

$$P_2(a \otimes b) = \sum S(a_{(1)}) S(a_{(3)}) b_{(1)} . (a_{(2)} \otimes b_{(2)}).$$

We first show that

$$\begin{aligned}
P_2(h.(a \otimes b)) &= \varepsilon(h) P_2(a \otimes b), \\
\Delta_s(h.(a \otimes b)) &= h. \Delta_s(a \otimes b).
\end{aligned}$$

Notice that

$$\begin{aligned}
&P_2(h.(a \otimes b)) \\
&= \sum P_2(h_{(1)}a \otimes h_{(2)}b) \\
&= \sum S(h_{(1)}a_{(1)}) S(h_{(3)}a_{(3)}) h_{(4)}b_{(1)} . (h_{(2)}a_{(2)} \otimes h_{(5)}b_{(2)}) \\
&= \sum S(h_{(1)}a_{(1)}) S(a_{(3)}) b_{(1)} . (h_{(2)}a_{(2)} \otimes h_{(3)}b_{(2)}) \\
&= \sum (S(h_{(1)}a_{(1)}) S(a_{(3)}) b_{(1)}) h_{(2)} . (a_{(2)} \otimes b_{(2)}) \\
&= \sum \varepsilon(h) S(a_{(1)}) S(a_{(3)}) b_{(1)} . (a_{(2)} \otimes b_{(2)}) \\
&= \varepsilon(h) P_2(a \otimes b).
\end{aligned}$$

So we get the first equation. For the second one,

$$\begin{aligned}
&\Delta_s(h.(a \otimes b)) \\
&= \Delta_s(\sum h_{(1)}a \otimes h_{(2)}b) \\
&= \sum h_{(1)}a_{(1)} \otimes h_{(2)}a_{(2)} S(h_{(4)}a_{(4)}) h_{(5)}b_{(1)} \otimes h_{(3)}a_{(3)} \otimes h_{(6)}b_{(2)} \\
&= \sum h_{(1)}a_{(1)} \otimes h_{(2)}a_{(2)} S(a_{(4)}) b_{(1)} \otimes h_{(3)}a_{(3)} \otimes h_{(4)}b_{(2)} \\
&= h. \sum a_{(1)} \otimes a_{(2)} S(a_{(4)}) b_{(1)} \otimes a_{(3)} \otimes b_{(2)} \\
&= h. \Delta_s(a \otimes b).
\end{aligned}$$

Then we have

$$\begin{aligned}
 & (P_2 \otimes P_2)\Delta_s(a \otimes b) \\
 &= \sum P_2(a_{(1)} \otimes a_{(2)}S(a_{(4)})b_{(1)}) \otimes P_2(a_{(3)} \otimes b_{(2)}) \\
 &= \sum P_2(a_{(1)} \cdot (1_H \otimes S(a_{(3)})b_{(1)})) \otimes P_2(a_{(2)} \otimes b_{(2)}) \\
 &= \sum \varepsilon(a_{(1)})P_2(1_H \otimes S(a_{(3)})b_{(1)}) \otimes P_2(a_{(2)} \otimes b_{(2)}) \\
 &= \sum P_2(1_H \otimes S(a_{(2)})b_{(1)}) \otimes P_2(a_{(1)} \otimes b_{(2)}) \\
 &= \sum S(S(a_{(3)})b_{(1)}) \cdot (1_H \otimes S(a_{(2)})b_{(2)}) \otimes P_2(a_{(1)} \otimes b_{(3)}),
 \end{aligned}$$

and

$$\begin{aligned}
 & \Delta_s P_2(a \otimes b) \\
 &= \Delta_s \left( \sum S(a_{(1)}S(a_{(3)})b_{(1)}) \cdot (a_{(2)} \otimes b_{(2)}) \right) \\
 &= \sum S(a_{(1)}S(a_{(3)})b_{(1)}) \cdot \Delta_s(a_{(2)} \otimes b_{(2)}) \\
 &= \sum S(a_{(1)}S(a_{(6)})b_{(1)}) \cdot (a_{(2)} \otimes a_{(3)}S(a_{(5)})b_{(2)} \otimes a_{(4)} \otimes b_{(3)}) \\
 &= \sum S(a_{(2)}S(a_{(7)})b_{(2)}) \cdot (a_{(3)} \otimes a_{(4)}S(a_{(6)})b_{(3)} \otimes S(a_{(1)}S(a_{(8)})b_{(1)}) \cdot (a_{(5)} \otimes b_{(4)}) \\
 &= \sum (S(a_{(2)}S(a_{(6)})b_{(2)})a_{(3)}) \cdot (1_H \otimes S(a_{(5)})b_{(3)}) \otimes S(a_{(1)}S(a_{(7)})b_{(1)}) \cdot (a_{(4)} \otimes b_{(4)}) \\
 &= \sum S(S(a_{(4)})b_{(2)}) \cdot (1_H \otimes S(a_{(3)})b_{(3)}) \otimes S(a_{(1)}S(a_{(5)})b_{(1)}) \cdot (a_{(2)} \otimes b_{(4)}).
 \end{aligned}$$

So

$$\begin{aligned}
 & (P_2 \otimes \text{id})\Delta_s P_2(a \otimes b) \\
 &= \sum \varepsilon(S(a_{(4)}))\varepsilon(b_{(2)})P_2(1_H \otimes S(a_{(3)})b_{(3)}) \otimes S(a_{(1)}S(a_{(5)})b_{(1)}) \cdot (a_{(2)} \otimes b_{(4)}) \\
 &= \sum P_2(1_H \otimes S(a_{(3)})b_{(2)}) \otimes S(a_{(1)}S(a_{(4)})b_{(1)}) \cdot (a_{(2)} \otimes b_{(3)}) \\
 &= \sum S(S(a_{(4)})b_{(2)}) \cdot (1_H \otimes S(a_{(3)})b_{(3)}) \otimes S(a_{(1)}S(a_{(5)})b_{(1)}) \cdot (a_{(2)} \otimes b_{(4)}) \\
 &= \Delta_s P_2(a \otimes b),
 \end{aligned}$$

and

$$\begin{aligned}
 & (\text{id} \otimes P_2)\Delta_s P_2(a \otimes b) \\
 &= \sum S(S(a_{(4)})b_{(2)}) \cdot (1_H \otimes S(a_{(3)})b_{(3)}) \otimes \varepsilon(a_{(1)}S(a_{(5)})b_{(1)})P_2(a_{(2)} \otimes b_{(4)}) \\
 &= \sum S(S(a_{(3)})b_{(1)}) \cdot (1_H \otimes S(a_{(2)})b_{(2)}) \otimes P_2(a_{(1)} \otimes b_{(3)}) \\
 &= (P_2 \otimes P_2)\Delta_s(a \otimes b).
 \end{aligned}$$

Therefore

$$(P_2 \otimes P_2)\Delta_s(a \otimes b) = (\text{id} \otimes P_2 + P_2 \otimes \text{id} - \text{id} \otimes \text{id})\Delta_s P_2(a \otimes b)$$

holds for any  $a, b \in H$ .

Finally,

$$\begin{aligned}
P_2^2(a \otimes b) &= P_2\left(\sum S(a_{(1)})S(a_{(3)})b_{(1)} \cdot (a_{(2)} \otimes b_{(2)})\right) \\
&= \sum \varepsilon(S(a_{(1)})S(a_{(3)})b_{(1)})P_2(a_{(2)} \otimes b_{(2)}) \\
&= \sum \varepsilon(a_{(1)})\varepsilon(a_{(3)})\varepsilon(b_{(1)})P_2(a_{(2)} \otimes b_{(2)}) \\
&= P_2(a \otimes b).
\end{aligned}$$

Then  $(H \otimes H, \Delta_s, \varepsilon_s, P_2)$  is an idempotent Rota-Baxter coalgebra of weight  $-1$ .

We mention here that the operators  $P_1$  and  $P_2$  are not of the type in Example 2.3 since they do not satisfy  $\Delta P = (P \otimes P)\Delta$ .

All examples given above are idempotent. Now we provide one which is not in such case.

**Example 2.5.** Let  $C$  be a  $\mathbb{K}$ -vector space with a basis  $\{c_n\}_{n \geq 0}$ . Define a  $\mathbb{K}$ -linear map  $\Delta : C \rightarrow C \otimes C$  by

$$\Delta(c_n) = \sum_{j=0}^n \sum_{i=n-j}^n (-1)^{i+j+n} \binom{n}{j} \binom{j}{n-i} c_i \otimes c_j,$$

and a  $\mathbb{K}$ -linear map  $\varepsilon : C \rightarrow \mathbb{K}$  by

$$\varepsilon(c_n) = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n > 0. \end{cases}$$

Let  $c_{-1} = 0$ . We define a  $\mathbb{K}$ -linear map  $P : C \rightarrow C$  by  $P(c_n) = c_{n-1}$ . Then  $(C, \Delta, \varepsilon, P)$  is a Rota-Baxter coalgebra of weight  $-1$ . The verification is given in the appendix.

### 3. BASIC PROPERTIES

We first discuss the relations between Rota-Baxter algebras and Rota-Baxter coalgebras. For a given vector space  $V$ , we denote by  $V^*$  the linear dual of  $V$ , i.e.,  $V^* = \text{Hom}_{\mathbb{K}}(V, \mathbb{K})$ . An endomorphism  $\varphi$  of  $V$  induces an endomorphism  $\varphi^*$  of  $V^*$  given by  $\varphi^*(f) = f\varphi$  for any  $f \in V^*$ . It is well-known that the linear dual of a coalgebra is an algebra. So we have

**Proposition 3.1.** *The linear dual of a (noncounitary) Rota-Baxter coalgebra together with the linear dual of the Rota-Baxter operator is a (nonunitary) Rota-Baxter algebra.*

*Proof.* It follows immediately from that

$$\begin{aligned}
\Delta^*(P^* \otimes P^*) &= ((P \otimes P)\Delta)^* \\
&= ((\text{id} \otimes P)\Delta P + (P \otimes \text{id})\Delta P + \lambda \Delta P)^* \\
&= P^* \Delta^*(\text{id} \otimes P^*) + P^* \Delta^*(P^* \otimes \text{id}) + \lambda P^* \Delta^*.
\end{aligned}$$

□

But the converse is not true in general, unless the dimension is finite. In order to overcome this difficulty we need some tools from the theory of Rota-Baxter algebras and that of coalgebras.

Let  $(R, P)$  be a Rota-Baxter algebra of weight  $\lambda$ . The *double product* is defined by

$$x \star_P y = xP(y) + P(x)y + \lambda xy, \quad \text{for } x, y \in R.$$

By the Theorem 1.1.17 in [15],  $R$  equipped with the product  $\star_P$  is a nonunitary  $\mathbb{K}$ -algebra which will be denoted by  $R_P$ , and  $(R, \star_P, P)$  is a Rota-Baxter algebra of weight  $\lambda$ . Moreover  $P$  is a homomorphism of algebras from  $R_P$  to  $R$ .

Recall that, for any unitary algebra  $(A, m, u)$ , the finite dual of  $A$  is

$$A^\circ = \{f \in A^* \mid \ker f \text{ contains a cofinite ideal of } A\}.$$

A cofinite ideal  $I$  of  $A$  is an ideal of  $A$  subject to  $\dim(A/I) < \infty$ . It is well-known that  $A^\circ$  is a coalgebra with comultiplication  $\Delta = m^*$  and counit  $\varepsilon = u^*$  (see [23]).

Under the notation above, we have the following result:

**Theorem 3.2.** *Let  $(R, P)$  be a unitary Rota-Baxter algebra of weight  $\lambda$ . If any ideal of  $R_P$  is also an ideal of  $R$ , then the finite dual  $(R^\circ, P^*)$  is a Rota-Baxter coalgebra of weight  $\lambda$ .*

*Proof.* For any cofinite ideal  $I$  of  $R_P$ ,  $I$  is also a cofinite ideal of  $R$  by the assumption, and hence  $R_P^\circ \subset R^\circ$ . Since  $P$  is an algebra map from  $R_P$  to  $R$ , we get  $P^*(R^\circ) \subset R_P^\circ$  by Lemma 6.0.1 (a) in [23]. Therefore  $P^*(R^\circ) \subset R^\circ$ . For any  $f \in R^\circ$  and  $a, b \in R$ , we have

$$\begin{aligned} \langle (P^* \otimes P^*)\Delta(f), a \otimes b \rangle &= \langle \Delta(f), P(a) \otimes P(b) \rangle = \langle f, P(a)P(b) \rangle \\ &= \langle f, P(P(a)b + aP(b) + \lambda ab) \rangle = \langle P^*(f), P(a)b + aP(b) + \lambda ab \rangle \\ &= \langle \Delta P^*(f), P(a) \otimes b + a \otimes P(b) + \lambda a \otimes b \rangle \\ &= \langle ((P^* \otimes \text{id})\Delta P^* + (\text{id} \otimes P^*)\Delta P^* + \lambda \Delta P^*)(f), a \otimes b \rangle, \end{aligned}$$

where the third equality holds since  $P$  is a Rota-Baxter operator. So the linear operator  $P^* : R^\circ \rightarrow R^\circ$  satisfies

$$(P^* \otimes P^*)\Delta = (P^* \otimes \text{id})\Delta P^* + (\text{id} \otimes P^*)\Delta P^* + \lambda \Delta P^*.$$

Hence the coalgebra  $(R^\circ, P^*)$  is a Rota-Baxter coalgebra of weight  $\lambda$ . □

**Example 3.3.** Let  $R = t\mathbb{K}[t]$  be the algebra of polynomials without constant terms and  $P$  the linear endomorphism of  $R$  given by  $P(t^n) = \frac{q^n}{1-q^n}t^n$ , where  $q \in \mathbb{K}$  is not a root of unity. Then  $(R, P)$  is a Rota-Baxter algebra of weight 1 (see Example 1.1.8 in [15]). From

$$\begin{aligned} t^n \star_P t^m &= P(t^n)t^m + t^n P(t^m) + t^{m+n} \\ &= \frac{q^n}{1-q^n}t^{m+n} + \frac{q^m}{1-q^m}t^{m+n} + t^{m+n} \\ &= \frac{1-q^{m+n}}{(1-q^n)(1-q^m)}t^{m+n}, \end{aligned}$$

we get

$$t^n t^m = t^n \star_P \frac{(1-q^n)(1-q^m)}{1-q^{m+n}} t^m.$$

So for any  $a \in R$ , there exists  $a' \in R$  such that  $t^n a = t^n \star_P a'$ . Furthermore, we can obtain that, for any  $x, a \in R$ , there exists  $a' \in R$  such that  $xa = x \star_P a'$ . Let  $I$  be an ideal of  $R_P$ . For any  $x \in I$  and  $a \in R$ , there exists  $a' \in R$  such that  $xa = x \star_P a'$ .

Since  $x \star_P a' \in I$ , then  $xa \in I$ . Hence  $I$  is also an ideal of  $R$ . So the finite dual  $(R^\circ, P^*)$  is a Rota-Baxter coalgebra of weight 1.

Let  $R$  be a locally finite graded algebra, i.e.,  $R$  has a direct decomposition  $R = \bigoplus_{i \geq 0} R_i$  of finite dimensional subspace  $R_i$  and  $R_i R_j \subset R_{i+j}$ . Then the graded dual  $R^g = \bigoplus_{i \geq 0} R_i^*$  is a coalgebra (see e.g., [23]). If moreover  $R$  is a Rota-Baxter algebra of weight  $\lambda$  with Rota-Baxter operator  $P$  which preserves the grading  $P(R_i) \subset R_i$ , then we have

**Proposition 3.4.** *The graded dual  $R^g$  of  $R$  is a Rota-Baxter coalgebra of weight  $\lambda$  together with the operator  $P^*$ .*

Now we turn to consider the relation between Rota-Baxter coalgebras and non-counitary ones. Obviously every Rota-Baxter coalgebra is noncounitary if we forget the counit. The reverse construction is not so easy. We provide a solution for the idempotent case in the following proposition.

**Proposition 3.5.** *Let  $(C, \Delta, P)$  be an idempotent noncounitary Rota-Baxter coalgebra of weight  $-1$ . Then there exists an idempotent Rota-Baxter coalgebra  $(\tilde{C}, \tilde{\Delta}, \varepsilon, \tilde{P})$  such that  $C = \ker \varepsilon$  as a subspace and the restriction of  $\tilde{P}$  on  $C$  is just  $P$ .*

*Proof.* We extend  $(C, \Delta)$  to a coalgebra  $(\tilde{C}, \tilde{\Delta}, \varepsilon)$  as follows. Fix a symbol  $\mathbf{1}$  and set  $\tilde{C} = \mathbb{K}\mathbf{1} \oplus C$  as a vector space. We define  $\tilde{\Delta}(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$  and  $\tilde{\Delta}(x) = \Delta(x) + \mathbf{1} \otimes x + x \otimes \mathbf{1}$  for any  $x \in C$ . The counit  $\varepsilon$  is defined to be the projection from  $\tilde{C}$  to  $\mathbb{K}\mathbf{1}$ . It is not hard to verify that  $(\tilde{C}, \tilde{\Delta}, \varepsilon)$  is a coalgebra. We extend  $P$  to  $\tilde{P}$  by setting that  $\tilde{P}(\mathbf{1}) = \mathbf{1}$  and  $\tilde{P}(x) = P(x)$  for  $x \in C$ . Then for any  $x \in C$ , we have

$$\begin{aligned}
& ((\text{id} \otimes \tilde{P})\tilde{\Delta}\tilde{P} + (\tilde{P} \otimes \text{id})\tilde{\Delta}\tilde{P} - \tilde{\Delta}\tilde{P})(\mathbf{1} + x) \\
&= ((\text{id} \otimes \tilde{P})\tilde{\Delta}\tilde{P} + (\tilde{P} \otimes \text{id})\tilde{\Delta}\tilde{P} - \tilde{\Delta}\tilde{P})(\mathbf{1}) \\
&\quad + ((\text{id} \otimes \tilde{P})\tilde{\Delta}\tilde{P} + (\tilde{P} \otimes \text{id})\tilde{\Delta}\tilde{P} - \tilde{\Delta}\tilde{P})(x) \\
&= \mathbf{1} \otimes \mathbf{1} + (\text{id} \otimes \tilde{P})\Delta\tilde{P}(x) + (\text{id} \otimes \tilde{P})(\mathbf{1} \otimes \tilde{P}(x) + \tilde{P}(x) \otimes \mathbf{1}) \\
&\quad + (\tilde{P} \otimes \text{id})\Delta\tilde{P}(x) + (\tilde{P} \otimes \text{id})(\mathbf{1} \otimes \tilde{P}(x) + \tilde{P}(x) \otimes \mathbf{1}) \\
&\quad - \Delta\tilde{P}(x) - \mathbf{1} \otimes \tilde{P}(x) - \tilde{P}(x) \otimes \mathbf{1} \\
&= \mathbf{1} \otimes \mathbf{1} + (\text{id} \otimes P)\Delta P(x) + \mathbf{1} \otimes P^2(x) + P(x) \otimes \tilde{P}(\mathbf{1}) \\
&\quad + (P \otimes \text{id})\Delta P(x) + \tilde{P}(\mathbf{1}) \otimes P(x) + P^2(x) \otimes \mathbf{1} \\
&\quad - \Delta P(x) - \mathbf{1} \otimes P(x) - P(x) \otimes \mathbf{1} \\
&= \mathbf{1} \otimes \mathbf{1} + (\text{id} \otimes P)\Delta P(x) + \mathbf{1} \otimes P(x) + P(x) \otimes \mathbf{1} \\
&\quad + (P \otimes \text{id})\Delta P(x) + \mathbf{1} \otimes P(x) + P(x) \otimes \mathbf{1} \\
&\quad - \Delta P(x) - \mathbf{1} \otimes P(x) - P(x) \otimes \mathbf{1} \\
&= \mathbf{1} \otimes \mathbf{1} + ((\text{id} \otimes P)\Delta P + (P \otimes \text{id})\Delta P - \Delta P)(x) \\
&\quad + \mathbf{1} \otimes P(x) + P(x) \otimes \mathbf{1} \\
&= \mathbf{1} \otimes \mathbf{1} + ((P \otimes P)\Delta)(x) + \mathbf{1} \otimes P(x) + P(x) \otimes \mathbf{1} \\
&= ((\tilde{P} \otimes \tilde{P})\tilde{\Delta})(\mathbf{1}) + ((\tilde{P} \otimes \tilde{P})\tilde{\Delta})(x)
\end{aligned}$$



$$= ((\tilde{P} \otimes \tilde{P})\tilde{\Delta})(1+x).$$

So  $(\tilde{C}, \tilde{\Delta}, \varepsilon, \tilde{P})$  is an idempotent Rota-Baxter coalgebra of weight  $-1$ .  $\square$

Suppose  $(C, \Delta, \varepsilon)$  is a coalgebra. A subspace  $J$  of  $C$  is called a *noncounitary coideal* if  $\Delta(J) \subset C \otimes J + J \otimes C$ . If moreover  $\varepsilon(J) = 0$ , then  $J$  is a coideal of  $C$ .

**Proposition 3.6.** *Let  $(C, \Delta, P)$  be a Rota-Baxter coalgebra of nonzero weight  $\lambda$ . Then  $P(C)$  is a noncounitary coideal of  $C$ . If  $P(C) \subset \ker \varepsilon$  then  $P(C)$  is a coideal of  $C$ . Furthermore, the quotient (noncounitary) coalgebra  $C/P(C)$  inherits a (noncounitary) Rota-Baxter coalgebra structure.*

*Proof.* By (1), we have

$$\Delta P = \frac{1}{\lambda} ((P \otimes P)\Delta - (\text{id} \otimes P)\Delta P - (P \otimes \text{id})\Delta P),$$

which implies that  $\Delta(P(C)) \subset C \otimes P(C) + P(C) \otimes C$ . The rest of the statements is straightforward.  $\square$

**Proposition 3.7.** *Let  $(C, \Delta, \varepsilon, P)$  be a Rota-Baxter coalgebra of weight  $\lambda$ . Define  $\overline{P} = -\lambda \text{id} - P$ . Then  $(C, \Delta, \varepsilon, \overline{P})$  is again a Rota-Baxter coalgebra of weight  $\lambda$ .*

*Proof.* By the definition, we have

$$\begin{aligned} & (\text{id} \otimes \overline{P})\Delta \overline{P} + (\overline{P} \otimes \text{id})\Delta \overline{P} + \lambda \Delta \overline{P} \\ &= (\text{id} \otimes (-\lambda \text{id} - P))\Delta(-\lambda \text{id} - P) + ((-\lambda \text{id} - P) \otimes \text{id})\Delta(-\lambda \text{id} - P) \\ & \quad + \lambda \Delta(-\lambda \text{id} - P) \\ &= \lambda^2 \Delta + \lambda \Delta P + \lambda(\text{id} \otimes P)\Delta + (\text{id} \otimes P)\Delta P \\ & \quad + \lambda^2 \Delta + \lambda \Delta P + \lambda(P \otimes \text{id})\Delta + (P \otimes \text{id})\Delta P - \lambda^2 \Delta - \lambda \Delta P \\ &= \lambda^2 \Delta + \lambda(\text{id} \otimes P)\Delta + \lambda(P \otimes \text{id})\Delta + (\text{id} \otimes P)\Delta P + (P \otimes \text{id})\Delta P + \lambda \Delta P \\ &= \lambda^2 \Delta + \lambda(\text{id} \otimes P)\Delta + \lambda(P \otimes \text{id})\Delta + (P \otimes P)\Delta \\ &= ((-\lambda \text{id} - P) \otimes (-\lambda \text{id} - P))\Delta \\ &= (\overline{P} \otimes \overline{P})\Delta. \end{aligned}$$

$\square$

Corresponding to the double product of Rota-Baxter algebra, we have the double coproduct construction.

**Proposition 3.8.** *Let  $(C, \Delta, P)$  be a noncounitary Rota-Baxter coalgebra of weight  $\lambda$ . We define  $\Delta_P = (P \otimes \text{id})\Delta + (\text{id} \otimes P)\Delta + \lambda \Delta$ . Then  $\Delta_P P = (P \otimes P)\Delta$  and  $(C, \Delta_P, P)$  is again a noncounitary Rota-Baxter coalgebra of weight  $\lambda$ .*

*Proof.* The equality

$$\Delta_P P = (P \otimes P)\Delta$$

follows from (1) immediately. We use it to show the coassociativity of  $\Delta_P$ :

$$\begin{aligned} & (\Delta_P \otimes \text{id})\Delta_P - (\text{id} \otimes \Delta_P)\Delta_P \\ &= (\Delta_P \otimes \text{id})(P \otimes \text{id} + \text{id} \otimes P + \lambda \text{id} \otimes \text{id})\Delta \\ & \quad - (\text{id} \otimes \Delta_P)(P \otimes \text{id} + \text{id} \otimes P + \lambda \text{id} \otimes \text{id})\Delta \end{aligned}$$

$$\begin{aligned}
&= (\Delta_P P \otimes \text{id} + \Delta_P \otimes P + \lambda \Delta_P \otimes \text{id}) \Delta \\
&\quad - (P \otimes \Delta_P + \text{id} \otimes \Delta_P P + \lambda \text{id} \otimes \Delta_P) \Delta \\
&= (((P \otimes P) \Delta) \otimes \text{id} + ((P \otimes \text{id} + \text{id} \otimes P + \lambda \text{id} \otimes \text{id}) \Delta) \otimes P \\
&\quad + ((\lambda P \otimes \text{id} + \lambda \text{id} \otimes P + \lambda^2 \text{id} \otimes \text{id}) \Delta) \otimes \text{id}) \Delta \\
&\quad - (P \otimes ((P \otimes \text{id} + \text{id} \otimes P + \lambda \text{id} \otimes \text{id}) \Delta) + \text{id} \otimes ((P \otimes P) \Delta) \\
&\quad + \lambda \text{id} \otimes ((P \otimes \text{id} + \text{id} \otimes P + \lambda \text{id} \otimes \text{id}) \Delta)) \Delta \\
&= (P \otimes P \otimes \text{id} + P \otimes \text{id} \otimes P + \text{id} \otimes P \otimes P + \lambda \text{id} \otimes \text{id} \otimes P \\
&\quad + \lambda P \otimes \text{id} \otimes \text{id} + \lambda \text{id} \otimes P \otimes \text{id} + \lambda^2 \text{id} \otimes \text{id} \otimes \text{id}) (\Delta \otimes \text{id}) \Delta \\
&\quad - (P \otimes P \otimes \text{id} + P \otimes \text{id} \otimes P + \lambda P \otimes \text{id} \otimes \text{id} + \text{id} \otimes P \otimes P \\
&\quad + \lambda \text{id} \otimes P \otimes \text{id} + \lambda \text{id} \otimes \text{id} \otimes P + \lambda^2 \text{id} \otimes \text{id} \otimes \text{id}) (\text{id} \otimes \Delta) \Delta \\
&= 0.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
&(\text{id} \otimes P) \Delta_P P + (P \otimes \text{id}) \Delta_P P + \lambda \Delta_P P \\
&= (\text{id} \otimes P)(P \otimes P) \Delta + (P \otimes \text{id})(P \otimes P) \Delta + \lambda (P \otimes P) \Delta \\
&= (P \otimes P)((P \otimes \text{id}) \Delta + (\text{id} \otimes P) \Delta + \lambda \Delta) \\
&= (P \otimes P) \Delta_P.
\end{aligned}$$

□

**Proposition 3.9.** *Let  $(C, \Delta)$  be a (noncounitary) coalgebra. A linear operator  $P$  on  $C$  is an idempotent Rota-Baxter operator of weight  $-1$  if and only if there is a  $\mathbb{K}$ -vector space direct sum decomposition  $C = C_1 \oplus C_2$  of  $C$  into noncounitary coideals  $C_1$  and  $C_2$  such that*

$$P : C \rightarrow C_1$$

*is the projection of  $C$  onto  $C_1$ :  $P(c_1 + c_2) = c_1$  for  $c_1 \in C_1$  and  $c_2 \in C_2$ .*

*Proof.* Suppose  $C$  has a  $\mathbb{K}$ -vector space direct sum decomposition  $C = C_1 \oplus C_2$ , where  $C_1$  and  $C_2$  are noncounitary coideals of  $C$ . Then the projection  $P$  of  $C$  onto  $C_1$  is idempotent since, for  $c = c_1 + c_2$  in  $C$  with  $c_1 \in C_1$  and  $c_2 \in C_2$ , we have

$$P^2(c) = P^2(c_1 + c_2) = P(c_1) = c_1 = P(c).$$

Further, for  $c = c_1 + c_2$  in  $C$  with  $c_1 \in C_1$  and  $c_2 \in C_2$ , we have  $\Delta(c_1) = \sum_i a'_i \otimes x_i + \sum_j y_j \otimes a''_j$  and  $\Delta(c_2) = \sum_i b'_i \otimes u_i + \sum_j v_j \otimes b''_j$ , where  $a'_i, a''_j \in C_1$ ,  $b'_i, b''_j \in C_2$  and  $x_i, y_j, u_i, v_j \in C$ . Then

$$\begin{aligned}
&(\text{id} \otimes P + P \otimes \text{id} - \text{id} \otimes \text{id}) \Delta(P(c)) \\
&= (\text{id} \otimes P + P \otimes \text{id} - \text{id} \otimes \text{id}) \Delta(c_1) \\
&= \sum_i a'_i \otimes P(x_i) + \sum_j y_j \otimes a''_j + \sum_i a'_i \otimes x_i + \sum_j P(y_j) \otimes a''_j \\
&\quad - \sum_i a'_i \otimes x_i - \sum_j y_j \otimes a''_j
\end{aligned}$$

$$\begin{aligned}
 &= \sum_i a'_i \otimes P(x_i) + \sum_j P(y_j) \otimes a''_j \\
 &= (P \otimes P)\Delta(c).
 \end{aligned}$$

Hence  $P$  is an idempotent Rota-Baxter operator of weight  $-1$ .

Conversely, suppose  $P : C \rightarrow C$  is an idempotent Rota-Baxter operator of weight  $-1$ . Let  $C_1 = P(C)$  and  $C_2 = (\text{id} - P)(C)$ . It is easy to see that  $C = C_1 \oplus C_2$  as vector spaces. Since  $c = P(c) + (\text{id} - P)(c)$  is the decomposition of  $c \in C$ , we see that  $P$  is the projection of  $C$  onto  $C_1$ . For any  $c_1 \in C_1$ , from

$$(P \otimes P)\Delta(c_1) = (\text{id} \otimes P + P \otimes \text{id} - \text{id} \otimes \text{id})\Delta(P(c_1)),$$

we have

$$\Delta(c_1) = (\text{id} \otimes P + P \otimes \text{id} - P \otimes P)\Delta(c_1) \in C \otimes C_1 + C_1 \otimes C.$$

So  $\Delta(C_1) \subset C \otimes C_1 + C_1 \otimes C$ .

For any  $c_2 \in C_2$ , we have

$$(P \otimes P)\Delta(c_2) = (\text{id} \otimes P + P \otimes \text{id} - \text{id} \otimes \text{id})\Delta(P(c_2)) = 0,$$

so  $\Delta(C_2) \subset \ker P \otimes C + C \otimes \ker P = C_2 \otimes C + C \otimes C_2$ . Hence  $C_1$  and  $C_2$  are noncounitary coideals of  $C$ .  $\square$

#### 4. APPENDIX

In this appendix, we present a detailed proof of Example 2.5. The techniques used here are similar to [3].

It is easy to see that  $\varepsilon$  is a counit of  $C$ . Before we check the coassociativity, we need the following lemma.

**Lemma 4.1.** *For integers  $n, k, l, j$  with  $0 \leq k, l, j \leq n$ , we have*

$$(2) \quad \sum_{i=0}^n \binom{j}{n-i} \binom{i}{l} \binom{l}{i-k} = \sum_{i=0}^n \binom{n-j}{n-i} \binom{i}{n-k} \binom{j}{i-l}.$$

*Proof.* For brevity, write the left-hand side and right-hand side of (2) as  $L(n, k, l, j)$  and  $R(n, k, l, j)$  respectively. We use induction on  $n$  with  $n \geq 0$ . When  $n = 0$ , then  $k = l = j = 0$ , we can check directly that

$$L(0, 0, 0, 0) = R(0, 0, 0, 0) = 1.$$

Assume that the equation  $L(n-1, k, l, j) = R(n-1, k, l, j)$  holds for any integers  $k, l, j, n$  with  $0 \leq k, l, j \leq n-1$ . Now consider the case for  $0 \leq k, l, j \leq n$ . First we prove some special cases. When  $k = n$ , we have

$$L(n, n, l, j) = \sum_{i=0}^n \binom{j}{n-i} \binom{i}{l} \binom{l}{i-n} = \binom{j}{0} \binom{n}{l} \binom{l}{0} = \binom{n}{l},$$

and

$$R(n, n, l, j) = \sum_{i=0}^n \binom{n-j}{n-i} \binom{j}{i-l} = \sum_{i=0}^n \binom{n-j}{i} \binom{j}{n-l-i}.$$

Using the classical Vandermonde's identity, for any integers  $x, y, z \geq 0$ ,

$$\sum_{t=0}^x \binom{x}{t} \binom{y}{z-t} = \binom{x+y}{z},$$

we get

$$R(n, n, l, j) = \binom{n}{n-l} = L(n, n, l, j).$$

When  $l = n$ , we have

$$L(n, k, n, j) = \sum_{i=0}^n \binom{j}{n-i} \binom{i}{n} \binom{n}{i-k} = \binom{j}{0} \binom{n}{n} \binom{n}{n-k} = \binom{n}{k},$$

and

$$R(n, k, n, j) = \sum_{i=0}^n \binom{n-j}{n-i} \binom{i}{n-k} \binom{j}{i-n} = \binom{n}{n-k} = L(n, k, n, j).$$

When  $j = n$ , since

$$\binom{i}{l} \binom{l}{i-k} = \binom{i}{k} \binom{k}{i-l} \text{ and } \binom{n}{i} \binom{i}{k} = \binom{n}{k} \binom{n-k}{n-i},$$

using the vandermonde's identity again, we have

$$L(n, k, l, n) = \sum_{i=0}^n \binom{n}{n-i} \binom{i}{k} \binom{k}{i-l} = \sum_{i=0}^n \binom{n}{k} \binom{n-k}{n-i} \binom{k}{i-l} = \binom{n}{k} \binom{n}{l},$$

and

$$R(n, k, l, n) = \sum_{i=0}^n \binom{0}{n-i} \binom{i}{n-k} \binom{n}{i-l} = \binom{n}{n-k} \binom{n}{n-l} = L(n, k, l, n).$$

When  $j = 0$ , we have

$$L(n, k, l, 0) = \sum_{i=0}^n \binom{0}{n-i} \binom{i}{l} \binom{l}{i-k} = \binom{n}{l} \binom{l}{n-k},$$

and

$$R(n, k, l, 0) = \sum_{i=0}^n \binom{n}{n-i} \binom{i}{n-k} \binom{0}{i-l} = \binom{n}{n-l} \binom{l}{n-k} = L(n, k, l, 0).$$

Next we will use the Pascal's rule

$$\binom{x}{y} = \binom{x-1}{y-1} + \binom{x-1}{y}$$

where  $x, y$  are integers with  $x \geq y \geq 0$ .

For  $0 \leq k, l < n$  and  $0 < j < n$ , we have

$$\begin{aligned} L(n, k, l, j) &= \sum_{i=0}^n \binom{j}{n-i} \binom{i}{l} \binom{l}{i-k} = \sum_{i=1}^n \binom{j}{n-i} \binom{i}{l} \binom{l}{i-k} \\ &= \sum_{i=1}^n \left[ \binom{j-1}{n-i} + \binom{j-1}{n-i-1} \right] \binom{i}{l} \binom{l}{i-k} \\ &= \sum_{i=1}^n \binom{j-1}{n-1-(i-1)} \binom{i-1+1}{l} \binom{l}{i-1-(k-1)} \\ &\quad + \sum_{i=1}^n \binom{j-1}{n-i-1} \binom{i}{l} \binom{l}{i-k} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i=0}^{n-1} \binom{j-1}{n-1-i} \binom{i+1}{l} \binom{l}{i-(k-1)} + \sum_{i=0}^{n-1} \binom{j-1}{n-1-i} \binom{i}{l} \binom{l}{i-k} \\
 &= \sum_{i=0}^{n-1} \binom{j-1}{n-1-i} \left[ \binom{i}{l-1} + \binom{i}{l} \right] \binom{l}{i-(k-1)} + L(n-1, k, l, j-1) \\
 &= \sum_{i=0}^{n-1} \binom{j-1}{n-1-i} \binom{i}{l-1} \left[ \binom{l-1}{i-k} + \binom{l-1}{i-(k-1)} \right] \\
 &\quad + L(n-1, k-1, l, j-1) + L(n-1, k, l, j-1) \\
 &= L(n-1, k, l-1, j-1) + L(n-1, k-1, l-1, j-1) \\
 &\quad + L(n-1, k-1, l, j-1) + L(n-1, k, l, j-1),
 \end{aligned}$$

and

$$\begin{aligned}
 R(n, k, l, j) &= \sum_{i=0}^n \binom{n-j}{n-i} \binom{i}{n-k} \binom{j}{i-l} = \sum_{i=1}^n \binom{n-j}{n-i} \binom{i}{n-k} \binom{j}{i-l} \\
 &= \sum_{i=1}^n \binom{n-1-(j-1)}{n-1-(i-1)} \binom{i-1+1}{n-1-(k-1)} \binom{j-1+1}{i-1-(l-1)} \\
 &= \sum_{i=0}^{n-1} \binom{n-1-(j-1)}{n-1-i} \binom{i+1}{n-1-(k-1)} \binom{j-1+1}{i-(l-1)} \\
 &= \sum_{i=0}^{n-1} \binom{n-1-(j-1)}{n-1-i} \left[ \binom{i}{n-1-k} + \binom{i}{n-1-(k-1)} \right] \\
 &\quad \times \left[ \binom{j-1}{i-l} + \binom{j-1}{i-(l-1)} \right] \\
 &= R(n-1, k, l, j-1) + R(n-1, k-1, l, j-1) + R(n-1, k, l-1, j-1) \\
 &\quad + R(n-1, k-1, l-1, j-1).
 \end{aligned}$$

By induction, we obtain  $L(n, k, l, j) = R(n, k, l, j)$ . This proves the lemma.  $\square$

Note that for integers  $x, y$ , we have

$$(3) \quad \binom{x}{y} = 0 \quad \text{if } x \geq 0 > y \text{ or } y > x \geq 0.$$

Then we can write

$$\Delta(c_n) = \sum_{j=0}^n \sum_{i=0}^n (-1)^{i+j+n} \binom{n}{j} \binom{j}{n-i} c_i \otimes c_j.$$

Now we are ready to check the coassociativity of  $(C, \Delta, \varepsilon)$ . We have

$$\begin{aligned}
 (\Delta \otimes \text{id})\Delta(c_n) &= \sum_{j=0}^n \sum_{i=0}^n (-1)^{i+j+n} \binom{n}{j} \binom{j}{n-i} \Delta(c_i) \otimes c_j \\
 &= \sum_{j=0}^n \sum_{i=0}^n (-1)^{i+j+n} \binom{n}{j} \binom{j}{n-i} \left( \sum_{l=0}^n \sum_{k=0}^n (-1)^{i+k+l} \binom{i}{l} \binom{l}{i-k} c_k \otimes c_l \right) \otimes c_j \\
 &= \sum_{j=0}^n \sum_{i=0}^n \sum_{l=0}^n \sum_{k=0}^n (-1)^{j+n+k+l} \binom{n}{j} \binom{j}{n-i} \binom{i}{l} \binom{l}{i-k} c_k \otimes c_l \otimes c_j,
 \end{aligned}$$

and

$$\begin{aligned}
(\text{id} \otimes \Delta)\Delta(c_n) &= \sum_{i=0}^n \sum_{k=0}^n (-1)^{k+i+n} \binom{n}{i} \binom{i}{n-k} c_k \otimes \Delta(c_i) \\
&= \sum_{i=0}^n \sum_{k=0}^n (-1)^{k+i+n} \binom{n}{i} \binom{i}{n-k} c_k \otimes \left( \sum_{j=0}^n \sum_{l=0}^n (-1)^{j+i+l} \binom{i}{j} \binom{j}{i-l} c_l \otimes c_j \right) \\
&= \sum_{i=0}^n \sum_{k=0}^n \sum_{j=0}^n \sum_{l=0}^n (-1)^{j+n+k+l} \binom{n}{i} \binom{i}{n-k} \binom{i}{j} \binom{j}{i-l} c_k \otimes c_l \otimes c_j \\
&= \sum_{i=0}^n \sum_{k=0}^n \sum_{j=0}^n \sum_{l=0}^n (-1)^{j+n+k+l} \binom{n}{j} \binom{n-j}{n-i} \binom{i}{n-k} \binom{j}{i-l} c_k \otimes c_l \otimes c_j.
\end{aligned}$$

The second equalities in the above two equations are consequences of the fact (3). The last equality follows from

$$\binom{n}{i} \binom{i}{j} = \binom{n}{j} \binom{n-j}{n-i}.$$

Due to Lemma 4.1, the coassociativity  $(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$  holds. Hence the coalgebra  $(C, \Delta, \varepsilon)$  is well-defined.

Next we check that  $P$  is a Rota-Baxter operator of weight  $-1$  on  $C$ . Since

$$\Delta P(c_n) = \Delta(c_{n-1}) = \sum_{j=0}^{n-1} \sum_{i=0}^{n-1} (-1)^{i+j+n-1} \binom{n-1}{j} \binom{j}{n-1-i} c_i \otimes c_j,$$

then

$$\begin{aligned}
&(\text{id} \otimes P + P \otimes \text{id} - \text{id} \otimes \text{id})\Delta P(c_n) \\
&= \sum_{j=1}^{n-1} \sum_{i=0}^{n-1} (-1)^{i+j+n-1} \binom{n-1}{j} \binom{j}{n-1-i} c_i \otimes c_{j-1} \\
&\quad + \sum_{j=0}^{n-1} \sum_{i=0}^{n-1} (-1)^{i+j+n-1} \binom{n-1}{j} \binom{j}{n-1-i} c_{i-1} \otimes c_j \\
&\quad - \sum_{j=0}^{n-1} \sum_{i=0}^{n-1} (-1)^{i+j+n-1} \binom{n-1}{j} \binom{j}{n-1-i} c_i \otimes c_j \\
&= \sum_{j=0}^{n-2} \sum_{i=0}^{n-1} (-1)^{i+j+n} \binom{n-1}{j+1} \binom{j+1}{n-1-i} c_i \otimes c_j \\
&\quad + \sum_{j=0}^{n-1} \sum_{i=0}^{n-2} (-1)^{i+j+n} \binom{n-1}{j} \binom{j}{n-2-i} c_i \otimes c_j \\
&\quad + \sum_{j=0}^{n-1} \sum_{i=0}^{n-1} (-1)^{i+j+n} \binom{n-1}{j} \binom{j}{n-1-i} c_i \otimes c_j \\
&= \sum_{j=0}^{n-1} \sum_{i=0}^{n-1} (-1)^{i+j+n} \left\{ \binom{n-1}{j+1} \left[ \binom{j}{n-1-i} + \binom{j}{n-2-i} \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& + \binom{n-1}{j} \binom{j}{n-2-i} + \binom{n-1}{j} \binom{j}{n-1-i} \} c_i \otimes c_j \\
& = \sum_{j=0}^{n-1} \sum_{i=0}^{n-1} (-1)^{i+j+n} \left[ \binom{n-1}{j+1} + \binom{n-1}{j} \right] \left[ \binom{j}{n-1-i} \binom{j}{n-2-i} \right] c_i \otimes c_j \\
& = \sum_{j=0}^{n-1} \sum_{i=0}^{n-1} (-1)^{i+j+n} \binom{n}{j+1} \binom{j+1}{n-1-i} c_i \otimes c_j \\
& = \sum_{j=0}^n \sum_{i=0}^n (-1)^{i+j+n} \binom{n}{j} \binom{j}{n-i} c_{i-1} \otimes c_{j-1} = (P \otimes P) \Delta(c_n).
\end{aligned}$$

Therefore  $(C, \Delta, \varepsilon, P)$  is a Rota-Baxter coalgebra of weight  $-1$ .

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